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2007 J. Phys. A: Math. Theor. 40 F51

(http://iopscience.iop.org/1751-8121/40/2/F01)

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INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) F51–F59 doi:10.1088/1751-8113/40/2/F01

FAST TRACK COMMUNICATION

Universal integrals for superintegrable systems on N-dimensional spaces of constant curvature

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Received 17 October 2006
Published 12 December 2006
Online at stacks.iop.org/JPhysA/40/F51

Abstract
An infinite family of classical superintegrable Hamiltonians defined on the N-dimensional spherical, Euclidean and hyperbolic spaces are shown to have a common set of $(2N-3)$ functionally independent constants of the motion. Among them, two different subsets of N integrals in involution (including the Hamiltonian) can always be explicitly identified. As particular cases, we recover in a straightforward way most of the superintegrability properties of the Smorodinsky–Winternitz and generalized Kepler–Coulomb systems on spaces of constant curvature and we introduce as well new classes of (quasi-maximally) superintegrable potentials on these spaces. Results presented here are a consequence of the $sl(2,\mathbb{R})$ Poisson coalgebra symmetry of all the Hamiltonians, together with an appropriate use of the phase spaces associated with Poincaré and Beltrami coordinates.

PACS numbers: 02.30.Ik, 02.40.Ky

1. Introduction

An N-dimensional (ND) completely integrable Hamiltonian $H^{(N)}$ is called *maximally superintegrable* (MS) if there exists a set of $(2N-2)$ globally defined functionally independent constants of the motion that Poisson-commute with $H^{(N)}$. Among these constants, at least, two different subsets of $(N-1)$ constants in involution can be found; these are connected with the fact that the system must possess separable solutions to the Hamilton–Jacobi equation in at least two different coordinate systems. From the dynamical point of view, all bounded motions of MS systems are closed and strictly periodic. In a similar way, if $H^{(N)}$ has $(2N-3)$ integrals with the above-mentioned properties the system will be called *quasi-maximally superintegrable* (QMS). Obviously, all MS systems are QMS ones.

The search of explicit MS (or QMS) systems is usually afforded by fixing the number N of degrees of freedom together with some assumptions concerning both the functional dependence of $H^{(N)}$ and of the integrals of the motion. In the case of natural systems with $H^{(N)} = T + U$, the configuration space of the system (a space with a given curvature, for instance) is chosen through a fixed expression for the kinetic energy T, and MS potentials U are investigated by imposing the existence of a suitable number of (unknown) integrals of the motion, that are usually assumed to be quadratic in the momenta and have also to be found simultaneously.

However, it turns out that although for a given space and dimension there exists a certain number of superintegrable potentials, only very few of them admit arbitrary ND generalizations. For example, in the ND Euclidean space the only two known MS Hamiltonians with integrals quadratic in the momenta are the oscillator potential with N centrifugal terms (the so-called Smorodinsky–Winternitz system [1, 2]) and the generalized Kepler–Coulomb potential with $(N-1)$ centrifugal terms [3].

The aim of this work is to face this problem from a quite different viewpoint, which is based on introducing a common $sl(2,\mathbb{R})\otimes\ldots^{N)}\otimes sl(2,\mathbb{R})$ symmetry framework for a large family of ND QMS systems with integrals quadratic in the momenta. Surprisingly enough, all the systems introduced in this way will share, by construction, the same set of $(2N-3)$ functionally independent integrals of the motion, which will be explicitly given. Among them, only one constant of the motion $C^{(N)}$ will depend on the N degrees of freedom and the remaining ones will be functions of a decreasing number of positions and momenta.

Therefore, in this framework the Hamiltonian $H^{(N)}$ will be the only function that characterizes each individual QMS system. In the case of natural systems, the kinetic term T will give us the information concerning the configuration space, and the potential term U will fully identify the interactions. We will show that, by following this approach, QMS systems on ND spaces of constant curvature can be easily understood and constructively studied. In this respect, the geometric interpretation of the canonical coordinates and momenta on each space turns out to be essential, and will be induced by the Hamiltonian transcription of the Poincaré and Beltrami coordinates.

Moreover, we stress that for some specific choices of $H^{(N)}$ the systems here presented are indeed MS ones, and we will recover as particular cases the ND Smorodinsky–Winternitz and generalized Kepler–Coulomb systems, that can now be understood as distinguished cases of an infinite family of systems with the same underlying symmetry. For both cases the additional independent integral of the motion, which is not provided by the underlying symmetry, can be found by other methods and will be explicitly given.

The paper is organized as follows. In the next section we present the main result that defines the infinite family of QMS systems together with the explicit form of their 'universal' $(2N-3)$ integrals of the motion. A sketch of the proof of this theorem is given, which is based on the symmetry of our systems under an $sl(2,\mathbb{R})\otimes\ldots^{N)}\otimes sl(2,\mathbb{R})$ algebra (or, alternatively, in terms of an $sl(2,\mathbb{R})$ Poisson coalgebra [4, 5]). Afterwards, many different examples of Hamiltonian systems belonging to this family (and thus sharing the same set of universal integrals) are presented. In section 3 we construct QMS systems on the ND Euclidean space $\mathbb{E}^N$ by using a kinetic energy T given in terms of the usual momenta conjugated to Cartesian coordinates, that can be naturally identified with the canonical coordinates $\mathbf{q}$. Some of these systems were already given in [5–7] but others are here identified within the coalgebra symmetry framework for the first time. Section 4 is devoted to QMS systems living on the ND spherical $\mathbb{S}^N$ and hyperbolic $\mathbb{H}^N$ spaces, for which the coordinates $\mathbf{q}$ are identified either with Poincaré or with Beltrami coordinates coming, respectively, from stereographic and central projections on $\mathbb{R}^{N+1}$. Finally, some remarks and open problems are mentioned.

2. An infinite family of QMS systems

The main result of this paper can be stated as follows.

Theorem 1. *Let* $\{\mathbf{q}, \mathbf{p}\} = \{(q_1, \ldots, q_N), (p_1, \ldots, p_N)\}$ *be N pairs of canonical variables. The ND Hamiltonian*

$$H^{(N)} = \mathcal{H}(\mathbf{q}^2, \tilde{\mathbf{p}}^2, \mathbf{q}\cdot\mathbf{p}) \tag{1}$$

with $\mathcal{H}$ *being any smooth function and*

$$\mathbf{q}^2 = \sum_{i=1}^{N} q_i^2 \qquad \tilde{\mathbf{p}}^2 = \sum_{i=1}^{N}\left(p_i^2 + \frac{b_i}{q_i^2}\right) \equiv \mathbf{p}^2 + \sum_{i=1}^{N}\frac{b_i}{q_i^2} \qquad \mathbf{q}\cdot\mathbf{p} = \sum_{i=1}^{N} q_i p_i, \tag{2}$$

where b_i *are arbitrary real parameters, is quasi-maximally superintegrable. The* $(2N-3)$ *functionally independent and 'universal' integrals of the motion are*

$$\begin{aligned} C^{(m)} &= \sum_{1\leqslant i<j}^{m}\left\{(q_i p_j - q_j p_i)^2 + \left(b_i\frac{q_j^2}{q_i^2} + b_j\frac{q_i^2}{q_j^2}\right)\right\} + \sum_{i=1}^{m} b_i \\ C_{(m)} &= \sum_{N-m+1\leqslant i<j}^{N}\left\{(q_i p_j - q_j p_i)^2 + \left(b_i\frac{q_j^2}{q_i^2} + b_j\frac{q_i^2}{q_j^2}\right)\right\} + \sum_{i=N-m+1}^{N} b_i, \end{aligned} \tag{3}$$

where $m = 2, \ldots, N$. *Moreover, the sets of N functions* $\{H^{(N)}, C^{(m)}\}$ *and* $\{H^{(N)}, C_{(m)}\}$ $(m = 2, \ldots, N)$ *are in involution.*

Some remarks are in order:

- Note that $C^{(N)} = C_{(N)}$, so that the number of different integrals is $(2N-3)$.
- The underlying symmetry can be interpreted as a generalization of the 'spherical' one, since $H^{(N)} = \mathcal{H}(\mathbf{q}^2, \mathbf{p}^2, \mathbf{q}\cdot\mathbf{p})$ is recovered when all $b_i = 0$. In this case, the constants of the motion (3) are just sums of squares of certain angular momentum components $l_{ij} = q_i p_j - q_j p_i$. In particular, $\mathcal{C}^{(m)} = \sum_{1\leqslant i<j}^{m} l_{ij}^2$ and $\mathcal{C}_{(m)} = \sum_{N-m+1\leqslant i<j}^{N} l_{ij}^2$.
- As we shall see in what follows, the additional centrifugal terms b_i/q_i^2 in $H^{(N)}$ come from the freedom to add such a term in the corresponding symplectic realization of an $sl(2,\mathbb{R})$ Poisson algebra. These terms will be essential in order to understand several known results concerning QMS systems on spaces of constant curvature.

Proof. This relies on the fact that, for any choice of the function $\mathcal{H}$, the Hamiltonian $H^{(N)}$ has an $sl(2,\mathbb{R})$ Poisson coalgebra symmetry [5] (under a certain symplectic realization) and, as a consequence, the integrals of the motion come from the left and right m-coproducts of the Casimir function for $sl(2,\mathbb{R})$ [6, 7]. Let us summarize the main steps of this construction.

We recall that the $sl(2,\mathbb{R})$ Poisson coalgebra is given by the following Lie–Poisson brackets, (primitive) coproduct Δ and Casimir:

$$\{J_3, J_+\} = 2J_+ \qquad \{J_3, J_-\} = -2J_- \qquad \{J_-, J_+\} = 4J_3 \tag{4}$$

$$\Delta(J_l) = J_l \otimes 1 + 1 \otimes J_l \qquad l = +, -, 3 \tag{5}$$

$$\mathcal{C} = J_- J_+ - J_3^2. \tag{6}$$

A one-particle symplectic realization of (4) is given by

$$J_- = q_1^2 \qquad J_+ = p_1^2 + b_1/q_1^2 \qquad J_3 = q_1 p_1, \tag{7}$$

where b_1 is a real parameter that labels the representation through $\mathcal{C} = b_1$. The coalgebra approach [5] provides the corresponding N-particle symplectic realization through the N-sites coproduct of (5) living on $sl(2,\mathbb{R}) \otimes \ldots^{N)} \otimes sl(2,\mathbb{R})$ [6]:

$$J_- = \sum_{i=1}^{N} q_i^2 \equiv \mathbf{q}^2 \quad J_+ = \sum_{i=1}^{N}\left(p_i^2 + \frac{b_i}{q_i^2}\right) \equiv \mathbf{p}^2 + \sum_{i=1}^{N}\frac{b_i}{q_i^2} \quad J_3 = \sum_{i=1}^{N} q_i p_i \equiv \mathbf{q}\cdot\mathbf{p}, \tag{8}$$

where b_i are N arbitrary real parameters. This means that the N-particle generators (8) fulfil the commutation rules (4) with respect to the canonical Poisson bracket

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right).$$

As a consequence of the coalgebra approach, these generators Poisson commute with the $(2N-3)$ functions (3) given by the sets $C^{(m)}$ and $C_{(m)}$, which are obtained from the 'left' and 'right' mth coproducts of the Casimir (6) with $m = 2, 3, \ldots, N$ (see [7] for details). If we label the N sites on $sl(2,\mathbb{R}) \otimes sl(2,\mathbb{R}) \otimes \cdots \otimes sl(2,\mathbb{R})$ by $1 \otimes 2 \otimes \cdots \otimes N$, the 'left' Casimir $\mathcal{C}^{(m)}$ is defined on the sites $1 \otimes 2 \otimes \cdots \otimes m$, while the 'right' one $\mathcal{C}_{(m)}$ is defined on $m \otimes \cdots \otimes N-1 \otimes N$. Moreover, it is straightforward to prove that the $2N-2$ functions $\{\mathcal{C}^{(2)}, \mathcal{C}^{(3)}, \ldots, \mathcal{C}^{(N)} \equiv \mathcal{C}_{(N)}, \mathcal{C}_{(N-1)}, \ldots, \mathcal{C}_{(2)}, \mathcal{H}\}$ are functionally independent (assuming that $\mathcal{H}$ is not a function of $\mathcal{C}$) and the coalgebra symmetry ensures that each of the subsets $\{\mathcal{C}^{(2)}, \ldots, \mathcal{C}^{(N)}, \mathcal{H}\}$ or $\{\mathcal{C}_{(2)}, \ldots, \mathcal{C}_{(N)}, \mathcal{H}\}$ is formed by N functions in involution [5, 7].

Therefore, any arbitrary function $\mathcal{H}$ defined on the N-particle symplectic realization of $sl(2,\mathbb{R})$ (8) is of the form (1), that is,

$$H^{(N)} = \mathcal{H}(J_-, J_+, J_3) = \mathcal{H}\left(\mathbf{q}^2, \mathbf{p}^2 + \sum_{i=1}^{N} \frac{b_i}{q_i^2}, \mathbf{q} \cdot \mathbf{p} \right)$$

and defines a QMS Hamiltonian system.

Note that when $N = 2$, the generic function $\mathcal{H}$ determines an *integrable* Hamiltonian as there is a single constant of the motion $\mathcal{C}^{(2)} \equiv \mathcal{C}_{(2)}$. In contrast, when $N = 3$ the Hamiltonian $\mathcal{H}$ (1) can be called minimally or 'weak' *superintegrable* (by following the terminology of [3] for this specific dimension) since it is endowed with three integrals $\{\mathcal{C}^{(2)}, \mathcal{C}^{(3)}, \mathcal{C}_{(2)}\}$. However, we remark that for arbitrary N there is a single constant of the motion left to assure maximal superintegrability. In this respect, we stress that some specific choices of $\mathcal{H}$ comprise maximally superintegrable systems as well, but the remaining integral does not come from the coalgebra symmetry and has to be deduced by making use of alternative procedures. □

3. QMS systems on the Euclidean space

It is immediate to realize that the ND kinetic energy on $\mathbb{E}^N$ directly arises through the generator J_+ (8) as it includes the usual momenta $\mathbf{p}^2$. Then if we add some smooth function $\mathcal{F}(J_-)$ as the potential term we can construct different QMS systems. Some possibilities are the following.

- *The ND Evans system*

$$\mathcal{H} = \frac{1}{2m} J_+ + \mathcal{F}(J_-) = \frac{1}{2m} \mathbf{p}^2 + \mathcal{F}(\mathbf{q}^2) + \sum_{i=1}^{N} \frac{\tilde{b}_i}{2q_i^2}, \tag{9}$$

  where $\tilde{b}_i = b_i/m$. This describes a particle of mass m under the action of a central potential $\mathcal{F}(\mathbf{q}^2)$ with N centrifugal terms associated with the $\tilde{b}_i$'s. The $N = 3$ case was introduced in [3].
- *The Smorodinsky–Winternitz system* [1, 2, 6, 8–10]. By choosing $\mathcal{F}(J_-) = \omega^2 J_-$ in the above Hamiltonian, we recover a system formed by an isotropic harmonic oscillator of mass m with angular frequency ω together with N centrifugal barriers:

$$\mathcal{H} = \frac{1}{2m} J_+ + \omega^2 J_- = \frac{1}{2m} \mathbf{p}^2 + \omega^2 \mathbf{q}^2 + \sum_{i=1}^{N} \frac{\tilde{b}_i}{2q_i^2}. \tag{10}$$

This Hamiltonian is known to be maximally superintegrable. A set of N additional constants of the motion (which do not come from the coalgebra symmetry) is

$$\mathcal{I}_i = p_i^2 + 2m\omega^2 q_i^2 + m\tilde{b}_i/q_i^2 \qquad i = 1, \ldots, N. \tag{11}$$

Each constant $\mathcal{I}_i$ is functionally independent with respect to both the set (3) and $\mathcal{H}$, thus it can be taken as the 'lost' integral.

- *A Garnier-type system* [11, 12]. If we set $\mathcal{F}(J_-) = \omega^2 J_- + \delta J_-^2$, with real parameter δ, we find a degenerate Garnier Hamiltonian (with $a_i = \omega^2$), which besides the previous harmonic term also comprises quartic oscillators:

$$\mathcal{H} = \frac{1}{2m} J_+ + \omega^2 J_- + \delta J_-^2 = \frac{1}{2m}\mathbf{p}^2 + \omega^2 \mathbf{q}^2 + \delta \mathbf{q}^4 + \sum_{i=1}^{N} \frac{\tilde{b}_i}{2q_i^2}. \tag{12}$$

The generalization to even-order nonlinear oscillators is straightforward, since the generic system

$$\mathcal{H} = \frac{1}{2m} J_+ + \omega^2 J_- + \sum_{k=1}^{\infty} \delta_k J_-^{k+1} = \frac{1}{2m}\mathbf{p}^2 + \omega^2 \mathbf{q}^2 + \sum_{k=1}^{\infty} \delta_k\, \mathbf{q}^{2(k+1)} + \sum_{i=1}^{N} \frac{\tilde{b}_i}{2q_i^2} \tag{13}$$

is a QMS one for any choice of the δ_k parameters.

- *A generalized Kepler–Coulomb system* [3]. The choice $\mathcal{F}(J_-) = -kJ_-^{-1/2}$, with real constant k, gives rise to the superposition of the Kepler–Coulomb potential with N centrifugal barriers:

$$\mathcal{H} = \frac{1}{2m} J_+ - kJ_-^{-1/2} = \frac{1}{2m}\mathbf{p}^2 - \frac{k}{\sqrt{\mathbf{q}^2}} + \sum_{i=1}^{N} \frac{\tilde{b}_i}{2q_i^2}. \tag{14}$$

Such a Hamiltonian is known to be maximally superintegrable under the condition that, at least, one of the centrifugal terms vanishes. In particular, if a single $\tilde{b}_i = 0$, an additional constant of the motion (and functionally independent with respect to (3) and (14)) arises. This additional integral is found to be

$$\mathcal{L}_i = \sum_{l=1}^{N} p_l(q_l p_i - q_i p_l) + km\frac{q_i}{\sqrt{\mathbf{q}^2}} - m\sum_{l=1;l\neq i}^{N} \tilde{b}_l \frac{q_i}{q_l^2}. \tag{15}$$

Likewise, if another $\tilde{b}_j = 0$ $(j \neq i)$, the function $\mathcal{L}_j$ can be proven to be also a constant of the motion, and so on. When all the $\tilde{b}_i$'s vanish the system reduces to the proper Kepler–Coulomb potential and the N additional constants of the motion $\mathcal{L}_i$ are the components of the Laplace–Runge–Lenz N-vector on $\mathbb{E}^N$.

- *Stationary electromagnetic fields*. Certain momenta-dependent potentials can also be obtained within this framework through an additional term depending on the generator J_3. Let us consider

$$\begin{aligned}\mathcal{H} &= \frac{1}{2m} J_+ - \frac{e}{m} J_3 \mathcal{G}(J_-) + e\mathcal{F}(J_-) \\ &= \frac{1}{2m}\mathbf{p}^2 - \frac{e}{m}(\mathbf{p}\cdot\mathbf{q})\mathcal{G}(\mathbf{q}^2) + e\mathcal{F}(\mathbf{q}^2) + \sum_{i=1}^{N} \frac{\tilde{b}_i}{2q_i^2},\end{aligned} \tag{16}$$

where e is a real parameter and $\mathcal{G}(J_-)$ is a smooth function. When $N = 3$, the above Hamiltonian describes a particle with mass m and charge e that moves on $\mathbb{E}^3$ under the action of an electromagnetic field, that is, $\mathcal{H} = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + e\psi$, where the (time-independent) vector $\mathbf{A}$ and scalar ψ potentials are given by

$$\psi(\mathbf{q}) = \mathcal{F}(\mathbf{q}^2) - \frac{e}{2m}\mathbf{q}^2\mathcal{G}^2(\mathbf{q}^2) + \sum_{i=1}^{3}\frac{\tilde{b}_i}{2eq_i^2} \qquad \mathbf{A}(\mathbf{q}) = \mathbf{q}\mathcal{G}(\mathbf{q}^2).$$

The electric $\mathbf{E} = -\nabla\psi$ and magnetic $\mathbf{H} = \nabla\times\mathbf{A}$ fields turn out to be

$$\mathbf{E} = \left(\frac{e}{m}\mathcal{G}^2 + \frac{2e}{m}\mathbf{q}^2\mathcal{G}\mathcal{G}' - 2\mathcal{F}'\right)\mathbf{q} + \frac{1}{e}\left(\frac{\tilde{b}_1}{q_1^3}, \frac{\tilde{b}_2}{q_2^3}, \frac{\tilde{b}_3}{q_3^3}\right) \qquad \mathbf{H} = 0,$$

where $\mathcal{G}'$ and $\mathcal{F}'$ are the derivatives with respect to their variable $\mathbf{q}^2$. Recall that 2D integrable electromagnetic Hamiltonians have been studied in [13–15]. We stress that this kind of construction can also be applied to obtain ND QMS Fokker–Planck Hamiltonians (see [13] and references therein).

- *Systems with coordinate-dependent mass*. If we multiply the former kinetic term J_+ in the above Hamiltonians by an arbitrary smooth function $\mathcal{M}(J_-)$, we obtain systems with a variable mass depending on $\mathbf{q}^2$ such as, for instance,

$$\mathcal{H} = \frac{1}{2\mathcal{M}(J_-)}J_+ + \mathcal{F}(J_-) = \frac{1}{2\mathcal{M}(\mathbf{q}^2)}\mathbf{p}^2 + \mathcal{F}(\mathbf{q}^2) + \sum_{i=1}^{N}\frac{b_i}{2\mathcal{M}(\mathbf{q}^2)q_i^2}. \tag{17}$$

We stress that expression (17) can be alternatively interpreted as a QMS system on ‘some’ space which would be determined through the metric coming from the kinetic energy $J_+/\mathcal{M}(J_-)$. In particular, in the next section we will show how certain specific Hamiltonians of the type (17) will define superintegrable systems on the sphere $\mathbb{S}^N$ and the hyperbolic space $\mathbb{H}^N$.

4. QMS systems on the sphere and the hyperbolic space

Recall that for a constant sectional curvature κ, both the ND sphere $\mathbb{S}^N$ ($\kappa > 0$) and the ND hyperbolic space $\mathbb{H}^N$ ($\kappa < 0$) can be embedded in an ambient linear space $\mathbb{R}^{N+1}$ with Weierstrass coordinates $(x_0, \mathbf{x}) = (x_0, x_1, \dots, x_N)$ fulfilling the ‘sphere’ constraint Σ: $x_0^2 + \kappa\mathbf{x}^2 = 1$. The metric on the proper ND spaces is given by [16, 17]:

$$\mathrm{d}s^2 = \frac{1}{\kappa}\left(\mathrm{d}x_0^2 + \kappa\,\mathrm{d}\mathbf{x}^2\right)\Big|_{\Sigma}, \tag{18}$$

where $\mathrm{d}\mathbf{x}^2 = \sum_{i=1}^{N}\mathrm{d}x_i^2$. The $N+1$ ambient coordinates can be parametrized in terms of N intrinsic coordinates in different ways.

Firstly, we consider the stereographic projection [18] from the ambient coordinates $(x_0, \mathbf{x}) \in \Sigma$ to the Poincaré ones $\mathbf{y} \in \mathbb{R}^N$ with pole $(-1, \mathbf{0}) \in \mathbb{R}^{N+1}$. Hence $(-1, \mathbf{0}) + \lambda(1, \mathbf{y}) \in \Sigma$ and we obtain that

$$\lambda = \frac{2}{1+\kappa\mathbf{y}^2} \qquad x_0 = \lambda - 1 = \frac{1-\kappa\mathbf{y}^2}{1+\kappa\mathbf{y}^2} \qquad \mathbf{x} = \lambda\mathbf{y} = \frac{2\mathbf{y}}{1+\kappa\mathbf{y}^2}. \tag{19}$$

Secondly, we apply the central projection from the Weiertrass coordinates to the Beltrami ones $\mathbf{z} \in \mathbb{R}^N$ with pole $(0, \mathbf{0}) \in \mathbb{R}^{N+1}$. Thus $(0, \mathbf{0}) + \mu(1, \mathbf{z}) \in \Sigma$ and we find

$$\mu = \frac{1}{\sqrt{1+\kappa\mathbf{z}^2}} \qquad x_0 = \mu \qquad \mathbf{x} = \mu\mathbf{z} = \frac{\mathbf{z}}{\sqrt{1+\kappa\mathbf{z}^2}}. \tag{20}$$

Then the metric (18) in Poincaré and Beltrami coordinates turns out to be

$$\mathrm{d}s^2 = 4\frac{\mathrm{d}\mathbf{y}^2}{(1+\kappa\mathbf{y}^2)^2} = \frac{(1+\kappa\mathbf{z}^2)\,\mathrm{d}\mathbf{z}^2 - \kappa(\mathbf{z}\cdot\mathrm{d}\mathbf{z})^2}{(1+\kappa\mathbf{z}^2)^2}. \tag{21}$$

Next if we write the metrics as Poincaré $\mathcal{T}^{\mathrm{P}}$ and Beltrami $\mathcal{T}^{\mathrm{B}}$ free Lagrangians

$$\mathcal{T}^{\mathrm{P}} = \frac{m\dot{\mathbf{y}}^2}{2(1+\kappa\mathbf{y}^2)^2} \qquad \mathcal{T}^{\mathrm{B}} = \frac{m}{2}\left(\frac{(1+\kappa\mathbf{z}^2)\dot{\mathbf{z}}^2 - \kappa(\mathbf{z}\cdot\dot{\mathbf{z}})^2}{(1+\kappa\mathbf{z}^2)^2}\right) \tag{22}$$

the momenta $\mathbf{p}_y$ and $\mathbf{p}_z$ conjugated to $\mathbf{y}$ and $\mathbf{z}$ can be deduced by

$$\mathbf{p}_y = \frac{m\dot{\mathbf{y}}}{(1+\kappa\mathbf{y}^2)^2} \qquad \mathbf{p}_z = m\left(\frac{(1+\kappa\mathbf{z}^2)\dot{\mathbf{z}} - \kappa(\mathbf{z}\cdot\dot{\mathbf{z}})\mathbf{z}}{(1+\kappa\mathbf{z}^2)^2}\right). \tag{23}$$

Therefore, the previous expressions show that the ND kinetic energy in both Poincaré $\mathcal{T}^{\mathrm{P}}$ and Beltrami $\mathcal{T}^{\mathrm{B}}$ phase spaces can be written as the following functions of the symplectic realization (8) of the generators of $sl(2,\mathbb{R})$:

$$\begin{aligned}\mathcal{T}^{\mathrm{P}} &= \frac{1}{2m}(1+\kappa J_-)^2 J_+ = \frac{1}{2m}(1+\kappa\mathbf{q}^2)^2\mathbf{p}^2\\ \mathcal{T}^{\mathrm{B}} &= \frac{1}{2m}(1+\kappa J_-)\left(J_+ + \kappa J_3^2\right) = \frac{1}{2m}(1+\kappa\mathbf{q}^2)(\mathbf{p}^2+\kappa(\mathbf{q}\cdot\mathbf{p})^2)\end{aligned} \tag{24}$$

provided that all $b_i = 0$, $(\mathbf{q},\mathbf{p}) \equiv (\mathbf{y},\mathbf{p}_y)$ for $\mathcal{T}^{\mathrm{P}}$ and $(\mathbf{q},\mathbf{p}) \equiv (\mathbf{z},\mathbf{p}_z)$ for $\mathcal{T}^{\mathrm{B}}$. When the b_i's are taken as arbitrary parameters, the N additional curved 'centrifugal' potentials arising in (24) are written in ambient coordinates in a very simple form:

$$\sum_{i=1}^{N}\frac{b_i}{2my_i^2}(1+\kappa\mathbf{y}^2)^2 = 2\sum_{i=1}^{N}\frac{\tilde{b}_i}{x_i^2} \qquad \sum_{i=1}^{N}\frac{b_i}{2mz_i^2}(1+\kappa\mathbf{z}^2) = \sum_{i=1}^{N}\frac{\tilde{b}_i}{2x_i^2}. \tag{25}$$

It is worth remarking that (25) is actually a direct consequence of the symplectic realization of $sl(2,\mathbb{R})$ (8) with $b_i \neq 0$. This, in turn, explains why the famous 'centrifugal' terms do appear everywhere in the classifications of superintegrable systems on the three classical Riemannian spaces [3, 19–21].

Recall as well that the radial (geodesic polar) distance r from an arbitrary point to the origin in $\mathbb{S}^N$ and $\mathbb{H}^N$ along the geodesic joining both points is expressed in terms of Weiertrass [16, 17], Poincaré and Beltrami coordinates as

$$\frac{1}{\kappa}\tan^2(\sqrt{\kappa}r) = \frac{\mathbf{x}^2}{x_0^2} = \frac{4\mathbf{y}^2}{(1-\kappa\mathbf{y}^2)^2} = \mathbf{z}^2. \tag{26}$$

Note that in the constant curvature analogues of the oscillator and Kepler–Coulomb problems the Euclidean radial distance is just replaced by the function $\frac{1}{\sqrt{\kappa}}\tan(\sqrt{\kappa}r)$.

Therefore, by taking into account (24)–(26) we finally present some QMS and MS Poincaré $\mathcal{H}^{\mathrm{P}}$ and Beltrami $\mathcal{H}^{\mathrm{B}}$ Hamiltonians that are constructed by adding some suitable functions depending on J_- to (24) and by considering arbitrary centrifugal terms b_i's. These systems are the curved counterpart of the Euclidean systems (9)–(15) which simultaneously cover $\mathbb{S}^N$ ($\kappa > 0$), $\mathbb{H}^N$ ($\kappa < 0$) and $\mathbb{E}^N$ ($\kappa = 0$; in this flat limit $r^2 = \mathbf{x}^2 = 4\mathbf{y}^2 = \mathbf{z}^2$). We stress that, again, all of them share the *same* set of constants of the motion (3), although in all these curved cases the geometric meaning of the canonical coordinates and momenta is completely different to the Euclidean one, as explained above. To make this geometrical interpretation more explicit, for each system we shall give both the Poincaré and Beltrami versions of each Hamiltonian.

- *A curved Evans system.* It would be given by

$$\begin{aligned}\mathcal{H}^{\mathrm{P}} &= \mathcal{T}^{\mathrm{P}} + \mathcal{F}\left(\frac{4J_-}{(1-\kappa J_-)^2}\right) = \frac{\left(1+\kappa\mathbf{q}^2\right)^2\mathbf{p}^2}{2m} + \mathcal{F}\left(\frac{4\mathbf{q}^2}{(1-\kappa\mathbf{q}^2)^2}\right) + \sum_{i=1}^{N}\frac{2\tilde{b}_i}{x_i^2}\\ \mathcal{H}^{\mathrm{B}} &= \mathcal{T}^{\mathrm{B}} + \mathcal{F}(J_-) = \frac{1}{2m}(1+\kappa\mathbf{q}^2)(\mathbf{p}^2+\kappa(\mathbf{q}\cdot\mathbf{p})^2) + \mathcal{F}(\mathbf{q}^2) + \sum_{i=1}^{N}\frac{\tilde{b}_i}{2x_i^2}.\end{aligned} \tag{27}$$

- *The curved Smorodinsky–Winternitz system* [16, 17]. Such a system is just the Higgs oscillator [22, 23] (that arises as either polynomial or rational potential in these coordinates) plus the corresponding centrifugal terms:

$$\begin{aligned}\mathcal{H}^{\mathrm{P}} &= \mathcal{T}^{\mathrm{P}} + \frac{4\omega^2 J_-}{(1-\kappa J_-)^2} = \frac{(1+\kappa\mathbf{q}^2)^2\mathbf{p}^2}{2m} + \frac{4\omega^2\mathbf{q}^2}{(1-\kappa\mathbf{q}^2)^2} + \sum_{i=1}^{N}\frac{2\tilde{b}_i}{x_i^2}\\ \mathcal{H}^{\mathrm{B}} &= \mathcal{T}^{\mathrm{B}} + \omega^2 J_- = \frac{1}{2m}(1+\kappa\mathbf{q}^2)(\mathbf{p}^2+\kappa(\mathbf{q}\cdot\mathbf{p})^2) + \omega^2\mathbf{q}^2 + \sum_{i=1}^{N}\frac{\tilde{b}_i}{2x_i^2}.\end{aligned} \tag{28}$$

This Hamiltonian is maximally superintegrable and the remaining constant of the motion can be taken from any of the following N functions (to be compared with (11)):

$$\begin{aligned}\mathcal{I}_i^{\mathrm{P}} &= (p_i(1-\kappa\mathbf{q}^2) + 2\kappa(\mathbf{q}\cdot\mathbf{p})q_i)^2 + \frac{8m\omega^2 q_i^2}{(1-\kappa\mathbf{q}^2)^2} + m\tilde{b}_i\frac{(1-\kappa\mathbf{q}^2)^2}{q_i^2}\\ \mathcal{I}_i^{\mathrm{B}} &= (p_i + \kappa(\mathbf{q}\cdot\mathbf{p})q_i)^2 + 2m\omega^2 q_i^2 + m\tilde{b}_i/q_i^2 \qquad i = 1,\ldots,N.\end{aligned} \tag{29}$$

- *A curved Garnier-type system*. By following the same prescription, QMS quartic curved oscillators can be defined as

$$\begin{aligned}\mathcal{H}^{\mathrm{P}} &= \mathcal{T}^{\mathrm{P}} + \frac{4\omega^2 J_-}{(1-\kappa J_-)^2} + \frac{16\delta J_-^2}{(1-\kappa J_-)^4}\\ &= \frac{(1+\kappa\mathbf{q}^2)^2\mathbf{p}^2}{2m} + \frac{4\omega^2\mathbf{q}^2}{(1-\kappa\mathbf{q}^2)^2} + \frac{16\delta\mathbf{q}^4}{(1-\kappa\mathbf{q}^2)^4} + \sum_{i=1}^{N}\frac{2\tilde{b}_i}{x_i^2}\\ \mathcal{H}^{\mathrm{B}} &= \mathcal{T}^{\mathrm{B}} + \omega^2 J_- + \delta J_-^2 = \frac{(1+\kappa\mathbf{q}^2)(\mathbf{p}^2+\kappa(\mathbf{q}\cdot\mathbf{p})^2)}{2m} + \omega^2\mathbf{q}^2 + \delta\mathbf{q}^4 + \sum_{i=1}^{N}\frac{\tilde{b}_i}{2x_i^2}.\end{aligned} \tag{30}$$

The expressions for the curved QMS analogues of the higher order nonlinear oscillators (13) are straightforward.

- *A curved generalized Kepler–Coulomb system* [19–21, 24–26]. The curved Kepler–Coulomb potential with N centrifugal terms corresponds to

$$\begin{aligned}\mathcal{H}^{\mathrm{P}} &= \mathcal{T}^{\mathrm{P}} - k\left(\frac{4J_-}{(1-\kappa J_-)^2}\right)^{-1/2} = \frac{(1+\kappa\mathbf{q}^2)^2\mathbf{p}^2}{2m} - k\frac{(1-\kappa\mathbf{q}^2)}{2\sqrt{\mathbf{q}^2}} + \sum_{i=1}^{N}\frac{2\tilde{b}_i}{x_i^2}\\ \mathcal{H}^{\mathrm{B}} &= \mathcal{T}^{\mathrm{B}} - kJ_-^{-1/2} = \frac{1}{2m}(1+\kappa\mathbf{q}^2)(\mathbf{p}^2+\kappa(\mathbf{q}\cdot\mathbf{p})^2) - \frac{k}{\sqrt{\mathbf{q}^2}} + \sum_{i=1}^{N}\frac{\tilde{b}_i}{2x_i^2}.\end{aligned} \tag{31}$$

This is again a MS system provided that, at least, one $\tilde{b}_i = 0$. In this case the remaining constant of the motion reads (compare with the Euclidean version (15)):

$$\begin{aligned}\mathcal{L}_i^{\mathrm{P}} &= \sum_{l=1}^{N}(p_l(1-\kappa\mathbf{q}^2) + 2\kappa(\mathbf{q}\cdot\mathbf{p})q_l)(q_l p_i - q_i p_l) + \frac{kmq_i}{2\sqrt{\mathbf{q}^2}} - m\sum_{l=1;l\neq i}^{N}\tilde{b}_l\frac{q_i(1-\kappa\mathbf{q}^2)}{q_l^2}\\ \mathcal{L}_i^{\mathrm{B}} &= \sum_{l=1}^{N}(p_l + \kappa(\mathbf{q}\cdot\mathbf{p})q_l)(q_l p_i - q_i p_l) + \frac{kmq_i}{\sqrt{\mathbf{q}^2}} - m\sum_{l=1;l\neq i}^{N}\tilde{b}_l\frac{q_i}{q_l^2}.\end{aligned} \tag{32}$$

If another $\tilde{b}_j = 0$, then $\mathcal{L}_j^{\mathrm{P,B}}$ is also a new constant of the motion. In this way the proper curved Kepler–Coulomb system [27] (with all the $\tilde{b}_i$'s equal to zero) is obtained, and in that case (32) are the N components of the Laplace–Runge–Lenz vector on $\mathbb{S}^N (\kappa > 0)$ and $\mathbb{H}^N (\kappa < 0)$.

More details on the latter Hamiltonians and their generalization to Lorentzian metrics will be given elsewhere. On the other hand, we remark that the definition of QMS systems on spaces of variable curvature can be achieved either by considering more general kinetic energy terms or by making use of coalgebra deformations, that have been already shown to underlie the superintegrability on some variable curvature (Riemannian and relativistic) spaces [28, 29]. Finally, the study of several interesting classes of non-natural Hamiltonians included in (1) is also worthy to be addressed in the future.

Acknowledgments

This work was partially supported by the Ministerio de Educación y Ciencia (Spain, Project FIS2004-07913) and by the Junta de Castilla y León (Spain, Project VA013C05).

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